Mathematics 222B Lecture 20 Notes

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1 Proof of Decay by Dispersion for the Wave Equation

1.1 Oscillatory integrals and the dispersive inequality for the wave equation

Last time, we were studying decay by dispersion for the wave equation, $\Box\phi = 0$. We saw that, using the Fourier transform, we could write the solution to the equation as

$$\phi(t,x) = \int a_+(\xi) e^{i(t|\xi| + x\cdot\xi)}\, d\xi + \int a_-(\xi) e^{i(-t|\xi| + x\cdot\xi)}\, d\xi.$$

The hope is that studying these integrals will allow us to prove our heuristically derived rate of dispersion for ϕ:

$$|\phi(t,x)| \lesssim \frac{1}{t^{(d-1)/2}}.$$

We studied the model oscillatory integral

$$I(\lambda) = \int a(\xi) e^{i\lambda\Phi(\xi)}\, d\xi$$

and proved two general principles:

Theorem 1.1 (Principle of non-stationary phase)**.** *If* $\operatorname{supp} a \subseteq \{|\partial_\xi \Phi| \geq \eta\}$*, then*

$$\left| \int a e^{i\lambda\Phi}\, d\xi \right| \lesssim_{k,\eta} \frac{1}{\lambda^k}$$

for all $k \geq 0$*.*

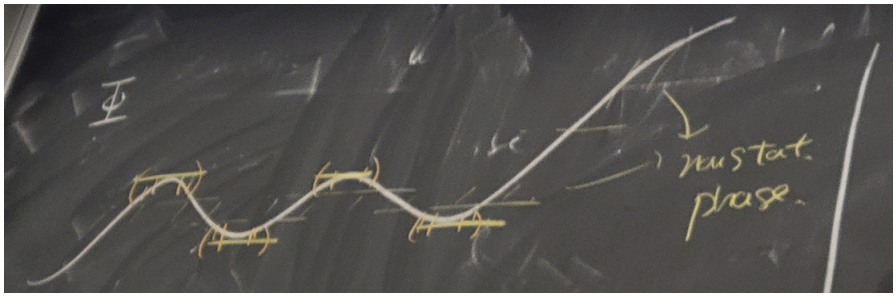
Theorem 1.2 (Principle of stationary phase)**.** *Suppose there exists one critical point of* Φ *(i.e. one zero of* $\partial_\xi \Phi$*) in* $\operatorname{supp} a$*. Then*

$$\left| \int a e^{i\lambda\Phi}\, d\xi \right| \lesssim_{\eta'} \operatorname{vol}(\{|\lambda\Phi| \leq \eta'\}).$$

In particular, if $\Phi = \xi^n$ *for* $n \geq 2$*, then*

$$|I(\lambda)| \lesssim \operatorname{vol}(\{\lambda|\xi|^n| \leq 1\}) \simeq \lambda^{1/n}.$$

Our justification for the principle of stationary phase was to use the dyadic decomposition. We chose this method because it is robust.



Now, let us return to the wave equation. Let's study

$$I(t,x) = \int a_+(\xi) e^{i(t|\xi| + x\cdot\xi)}\, d\xi,$$

where a_+ is the amplitude and $t|\xi| + x \cdot \xi$ is the phase.

Definition 1.1. Let the **Besov norm** be defined as

$$\|f\|_{B_r^{s,p}} := \left(\sum_k (2^{sk} \|P_k f\|_{L^p})^r \right)^{1/r},$$

where P_k is the Littlewood-Paley projection

$$\widehat{P}_k f = \chi_0(\xi/2^k)\widehat{f}(\xi)$$

with $\operatorname{supp} \chi_0(\cdot/2^k) \subseteq \{|\xi| \simeq 2^k\}$ and $\sum_{k=-\infty}^{\infty} \chi_0(\xi/2^k) = 1$ for $\xi \neq 0$.

Theorem 1.3 (Dispersive inequality for the wave equation)**.** *Consider a solution ϕ to the wave equation*

$$\begin{cases} \Box\phi = 0 \\ (\phi, \partial_t\phi|_{t=0}(g,h). \end{cases}$$

Then

$$\|\phi(t,x)\|_{L_x^\infty} \lesssim t^{-(d-1)/2} (\|g\|_{B_1^{\frac{d+1}{2},1}} + \|h\|_{B_1^{\frac{d-1}{2},1}}).$$

The $\frac{d+1}{2}, \frac{d-1}{2}$ can be determined by dimensional analysis.

1.2 Reduction to an oscillatory integral with projected amplitude

In general, if we want $L^1 \to L^\infty$-type bounds, it usually suffices to just consider fundamental solutions; the idea is that any L^1 data can be split into delta distributions by convolution. A fundamental solution E_+ to the wave equation (with initial data $g = 0$ and $\phi = E_+ * h$) is

$$\begin{cases} \Box E_+ = 0 & t > 0 \\ (E_t, \partial_t E_t)|_{t=0} = (0, \delta_0). \end{cases}$$

In Fourier space, the initial data looks like

$$(\widehat{E}_+, \partial_t \widehat{E}_+)|_{t=0} = (0, 1).$$

the constant 1 function has non-compact support, so we want to use a cutoff.

Instead, think of $P_k E_+$, which is the solution to the equation with initial data

$$(\widehat{P_k E_+}, \partial_t \widehat{P_k E_+})|_{t=0} = (0, \chi_0(\xi/2^k)),$$

which will give us a nicer oscillatory integral. This will be enough because we can decompose

$$\begin{aligned} \phi &= E_+ * h \\ &= \sum_k ((P_k E_+) * h\delta_{t=0}) \\ &= \sum_k (\widetilde{P}_k P_k E_+) * h\delta_{t=0}, \end{aligned}$$

where $\widetilde{P}_k$ has the same properties as P_k but with $\widetilde{P}_k P_k = 1$. (We saw this in our study of Schauder theory.)

$$= \sum_k (P_k E_+ * \widetilde{P}_k h\delta_{t=0}).$$

We claim that it suffices to prove that

$$\|P_k E_+\|_{L^\infty_x} \lesssim t^{-\frac{d-1}{2}} \|\check{\chi}_0(\cdot/2^k)\|_{L^1} 2^{k\frac{d-1}{2}}.$$

Proof. If this bound holds, then

$$\begin{aligned} \|\phi(t,x)\|_{L^\infty} &= \left\| \sum_k \int P_k E_+(t, x-y) \widetilde{P}_k h(y)\, dy \right\|_{L^\infty} \\ &\lesssim \sum_k \int \|P_k E_+(t, x-y)\|_{L^\infty} |\widetilde{P}_k h(y)|\, dy \\ &\lesssim t^{-\frac{d-1}{2}} \sum_k 2^{k\frac{d-1}{2}} \|\widetilde{P}_k h\|_{L^1}. \end{aligned}$$

□

We now claim that it suffices to take $k = 0$. This is because out bound is invariant under the scaling $(t, x) \mapsto (\lambda t, \lambda x)$. This means that we only need to prove

$$\|P_0 E_+\|_{L^\infty} \lesssim t^{-\frac{d-1}{2}}.$$

$P_0 E_+$ is an oscillatory integral of the form

$$P_0 E_+ = \int a_+(\xi) e^{i(t|\xi| + x\cdot\xi)}\, dx + \int a_-(\xi) e^{i(-t|\xi| + x\cdot\xi)},$$

where $a_\pm$ have support in $\{|\xi| \simeq 1\}$ and obey $|D^\alpha a_\pm| \lesssim_\alpha 1$.

Hence, it suffices to consider

$$I(t, x) = \int a_+(\xi) e^{i(t|\xi| + x\cdot\xi)}\, dx \underbrace{\lesssim t^{-\frac{d-1}{2}}}_{\text{want}}$$

with $\operatorname{supp} a_+ \subseteq \{|\xi| \simeq 1\}$ and $|D^\alpha a_+| \lesssim_\alpha 1$.

1.3 Estimating the size of the oscillatory integrals

To estimate the size of this oscillatory integral, we look at the critical points of the phase

$$\Phi = t|\xi| + x \cdot \xi.$$

When is $\nabla\Phi = 0$? Observe that we have the identity $\partial_{\xi_j} e^{i\Phi} i \partial_{\xi_j} \Phi e^{i\Phi}$, so

$$e^{i\Phi} = \frac{1}{i\partial_{\xi_j}\Phi} e^{i\Phi}.$$

We may assume, by rotation in x-space that x is parallel to the vector e_1. Then

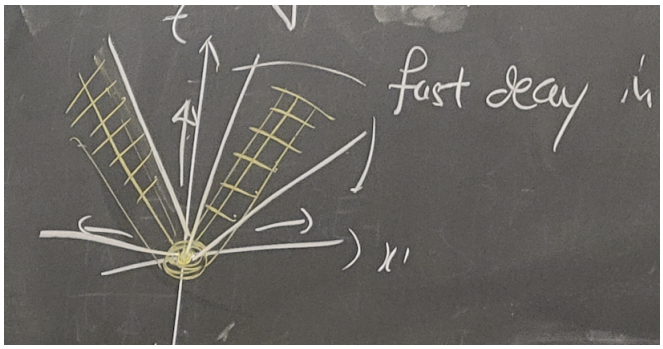
$$\Phi = t|\xi| + x^1\xi_1, \qquad \partial_{\xi_1} = \frac{t\xi_1}{|\xi|} + x^1, \qquad \partial_{\xi_j}\Phi = t\frac{\xi_j}{|\xi|}.$$

for $j \neq 1$. Then $\xi_j = 0$ for $j \neq 1$ occurs when $\frac{\xi_1}{|\xi|} = -\frac{x^1}{t}$. But $\xi_j = 0$ for $j \neq 1$ implies that $|\xi_1| = |\xi|$, so

$$\{\nabla\Phi = 0\} = \begin{cases} \varnothing & \text{if } |\frac{x^1}{t}| \neq 1 \\ \{s(-\frac{x^1}{t}, 0, \ldots, 0) : s > 0\} & \text{if } |\frac{x^1}{t}| = 1. \end{cases}$$

If $|\frac{x^1}{t}| > c$ or $|\frac{x^1}{t}| < \frac{1}{c}$, then we the principle of non-stationary phase should apply, and we should be able to get $\frac{1}{\max(|t|,|x|)^k}$. The fundamental solution is a cone, and we smoothed

it out with the projection. This says that ifwe look at a cone inside or outside this original cone, we get fast decay in t and $|x|$.



Assume that $|\frac{x^1}{t}| \simeq 1$. We need to look at the domain of Φ near the critical points

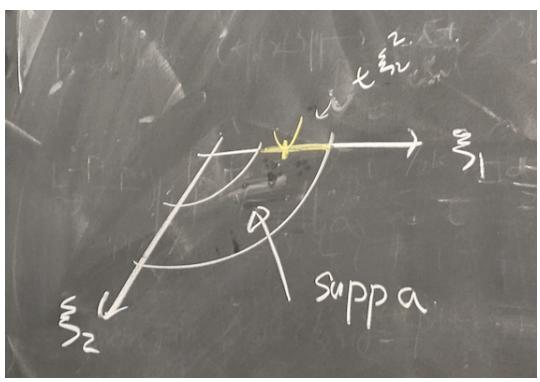
$$\partial_k \Phi = t\frac{\xi_k}{|\xi|} + x^1 \delta 1, k,$$

$$\partial_{\xi_j}\partial_{\xi_k}\Phi = -t\frac{\xi_j\xi_k}{|\xi|^3} + t\frac{\delta_{j,k}}{|\xi|} = t\frac{|\xi|^2\delta_{j,k} - \xi_j\xi_k}{|x|^3}.$$

At a critical point, $\xi = (-s\frac{x^1}{t}, 0, \ldots, 0)$,

$$\nabla^2\Phi = \begin{bmatrix} 0 & 0 \\ 0 & \frac{t}{|\xi|}I \end{bmatrix}.$$

And remember that on the support of a, $|\xi| \simeq 1$. Here is the picture:



So the region of stationary phase is $\{|t(\xi_2^2 + \cdots \xi_d^2)| \lesssim 1\}$, and

$$\mathrm{vol}_{\xi_2,\ldots,\xi_d}(\{|t(\xi_2^2 + \cdots \xi_d^2)| \lesssim 1\}) \lesssim t^{-\frac{d-1}{2}}.$$

By the principle of stationary phase, $t^{-\frac{d-1}{2}}$ dictates the size of $I(t,x)$.

The actual result can be proven via dyadic decomposition into regions of the form $\{t|\xi'|^2 \simeq \alpha\}_{\alpha=2^0,2^1,\ldots}$, where $\xi' = (\xi_2, \ldots, \xi_d)$.